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# On the relation between modular theory and geometry

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**Abstract**

In this paper we comment on part of a recent paper by Schroer and Wiesbrock. Therein they calculate some new modular structure for the $U(1)$-current algebra (Weyl algebra). We point out that their findings are true in a more general setting. The split-property allows an extension to doubly-localized algebras.

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## 1. Introduction

We would like to add a point to a recent inspiring work of Schroer and Wiesbrock [1][1] implying a modular origin of the chiral 'higher dilation' diffeomorphisms.

For the $U(1)$-current algebra in two-dimensional spacetime they construct states invariant under higher representations of the Möbius group, generated by the modes $L_{-n,0,n}$. The new states fulfil the KMS-property with respect to modified dilations. These modified dilations are identified with the modular group associated with von Neumann algebras localized in well-chosen regions and the new states.

In the original version [1] of the conjecture it was overlooked that the chosen state lacked the property of faithfulness if used on doubly-localized intervals. As far as the demonstration of the modular origin of the diffeomorphism group was concerned this was corrected in the work of Schroer and Fassarella [3], but the correction was at the expense of the original conjecture. Here, we show that due to the geometrical properties of the modified transformations, respectively the new F-S states, the use of the split-property [16] allows a faithful extension of these F-S states to doubly-localized intervals and in this way the demonstration of the modular origin of the diffeomorphism group is harmonized with the understanding of the modular structure of double intervals.

One can also show that their calculations in the $U(1)$ case extend to general rational conformal field theories.

[1] Recently, this work was corrected in an important point and also extended by Schroer and Fassarella [3]. Their new results also demanded a correction of an earlier version of this paper.

In section 2, we sketch briefly the ansatz and the result concerning the modular structure in the case of the $U(1)$-current algebra. Section 3 contains our point to add, providing a more general point of view of the aforementioned results of Schroer and Wiesbrock. Their findings are shown to be true in the general setting of a rational conformal field theory. As a non-trivial example it serves the theory of non-Abelian currents. In section 4, we investigate so-called multilocalized algebras by using the split-property. The product states and modular group of such algebras are identified. Some additional remarks and a short summary are given in section 5.

## 2. Schroer–Wiesbrock setting: *U*(1)-current algebra

Conformal field theory in two dimensions ($CFT_2$) [15] provides a well-suited realm for algebraic quantum field theory [2][2], especially for problems concerning the geometric identification of the modular structure [4, 5].

Minkowskian $CFT_2$ may be represented on the product of two circles, $S^1 \times S^1$-spacetime (the 'compact picture'). The global symmetry group of the $CFT_2$ is the Möbius group $PSU(1,1) \times PSU(1,1)$. We will concentrate on one of the groups, being realized on one of the circles:

$$PSU(1,1) := SU(1,1)/\{\pm 1\}$$

$$SU(1,1) := \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

The spectrum generating algebra of reparametrizations of the circle is generated by the Virasoro algebra (with central charge $c$) $\mathfrak{L}_c$:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \qquad n \in \mathbb{Z}. \tag{1}$$

The globally realized group $PSU(1,1)$ has the underlying generators $L_{-1}, L_0, L_1$, fulfilling a $sl(2,\mathbb{C})$-algebra:

$$[L_1, L_{-1}] = 2L_0 \qquad [L_{\pm 1}, L_0] = \pm L_{\pm 1}. \tag{2}$$

The Virasoro algebra $\mathfrak{L}_c$ contains infinitely many further $sl(2,\mathbb{C})$-algebras, generated by the modes $L_{-n}, L_0, L_n, n > 1$:

$$\begin{aligned} &L_{-n} \mapsto \tilde{L}_{-n} := \frac{1}{n} L_{-n} \\ &L_0 \mapsto \tilde{L}_0 := \frac{1}{n} L_0 + \frac{c}{24}\frac{(n^2-1)}{n} \longmapsto \left\{ \begin{aligned} [\tilde{L}_{+n}, \tilde{L}_{-n}] &= 2\tilde{L}_0 \\ [\tilde{L}_{\pm n}, \tilde{L}_0] &= \pm \tilde{L}_{\pm n} \end{aligned} \right\} sl(2,\mathbb{C}). \\ &L_{+n} \mapsto \tilde{L}_{+n} := \frac{1}{n} L_{+n} \end{aligned} \tag{3}$$

The corresponding finite transformations are of the form:

$$g_n(z) := \left( \frac{\alpha z^n + \beta}{\bar{\beta} z^n + \bar{\alpha}} \right)^{\frac{1}{n}} \qquad \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in PSU(1,1). \tag{4}$$

They leave the unit-circle $S^1$ invariant.

One may equally well represent the $CFT_2$ on a product of lines, $\mathbb{R} \times \mathbb{R}$-spacetime (the 'non-compact picture'). The coordinate transformation from the circle to the line is provided by the stereographic projection (Cayley transformation):

$$S^1 \backslash \{-1\} \ni z \longmapsto x(z) := -\mathrm{i}\frac{z-1}{z+1} \in \mathbb{R} \qquad -1 \longmapsto \infty. \tag{5}$$

[2] For a very recent review of the current state of algebraic QFT, see [6].

The global symmetry group $PSU(1,1)$ transforms isomorphically in this process into the real group $PSL(2,\mathbb{R})$ [7]:

$$\mathbb{R} \ni x \longmapsto \hat{g}(x) := \frac{ax+b}{cx+d} \qquad ad - bc = 1.$$

It is slightly more cumbersome to handle the transformations analogue to equation (4) in the non-compact picture. For this reason we perform the calculations in the compact picture representation.

Schroer and Wiesbrock take as a paradigm of their discussion the $U(1)$-current algebra on the circle. The constituting relation of this $U(1)$-algebra is the current–current commutation relation (with the circle as base space):

$$[J(z), J(w)] = -\partial_z \delta(z-w).$$

In order to bring the algebraic ansatz [16] to the stage one has to smear the currents by real testfunctions on the circle:

$$J(f) := \int_{S^1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} f(z)\, J(z).$$

The bounded operators $\mathrm{e}^{\mathrm{i}J(f)}$ give rise to a von Neumann algebra, i.e., weakly closed algebra of bounded operators:

$$\mathcal{W}(\mathcal{I}) := \{W(f) := \mathrm{e}^{\mathrm{i}J(f)} |\, \mathrm{supp}(f) \subseteq \mathcal{I} \subset S^1\}''$$

which is called (local) Weyl algebra [8, 17]. The double prime indicates here the double commutant which after a theorem by von Neumann (double-commutant theorem [17]) equals the weak closure of the algebra generated by the $\mathrm{e}^{\mathrm{i}J(f)}$. The net of Weyl algebras $\{S^1 \supset \mathcal{I} \mapsto \mathcal{W}(\mathcal{I})\}$ fulfils the postulates of algebraic quantum field theory, in particular, locality [10].

The one-parameter group of dilations in $PSU(1,1)$ has the following form:

$$\mathrm{Dil}(t)z = \frac{\mathrm{ch}(\pi t)z + \mathrm{sh}(\pi t)}{\mathrm{sh}(\pi t)z + \mathrm{ch}(\pi t)} \qquad t \in \mathbb{R} \qquad z \in S^1. \tag{6}$$

These mappings have the points $\{1, -1\} \in S^1$ as fixpoints. The upper and lower semi-circles, $S^1_+$ and $S^1_-$, respectively are mapped by dilations onto themselves. A dilation group attached to an arbitrary, proper interval $\mathcal{I} \subset S^1$ (mapping this interval onto itself) is constructed as follows:

$$PSU(1,1) \ni \mathrm{Dil}_{\mathcal{I}}(t) := g_{\mathcal{I}}^{-1} \mathrm{Dil}(t) g_{\mathcal{I}} \qquad g_{\mathcal{I}} \in PSU(1,1) \quad g_{\mathcal{I}}\mathcal{I} = S^1_+. \tag{7}$$

The interval $\mathcal{I}$ is mapped bijectively to the upper semi-circle, dilated and mapped back as one can see in the following diagram:

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{\mathrm{Dil}_{\mathcal{I}}} & \mathcal{I} \\ {\scriptstyle g_{\mathcal{I}}}\downarrow & & \uparrow{\scriptstyle g_{\mathcal{I}}^{-1}} \\ S^1_+ & \xrightarrow[\mathrm{Dil}]{} & S^1_+ \end{array}.$$

The representation of the one-parameter group of dilations (equation (7)) gives (a geometric realization of) the unique modular group $\left\{\Delta^{it}_{\mathcal{I}}; \, t \in \mathbb{R}\right\}$ [16, 17] of the (vacuum) tuple $(\mathcal{W}(\mathcal{I}), \omega_0)$. For the case of the upper and lower semi-circles which in the non-compact picture become positive and negative lightrays this follows from the work of Bisognano and Wichmann [4]. It is a peculiarity of $\mathrm{CFT}_2$ that arbitrary intervals can be mapped onto the upper (or lower) semi-circle which therefore allows to identify the modular group of algebras localized in these regions with the above constructed dilations.

The vacuum expectation values of Weyl operators obey the KMS-condition [11, 16, 17] with respect to (the representation of) the one-parameter group of dilations [12]:

$$\omega_0\big(W(f)\ Ad\left[U_{\mathrm{Dil}_{\mathcal{I}}(t)}\right](W(g))\big) \overset{\mathrm{KMS}}{=} \omega_0\left(Ad\left[U_{\mathrm{Dil}_{\mathcal{I}}(t+i)}\right](W(g))\ W(f)\right) \tag{8}$$

$W(f),\ W(g) \in \mathcal{W}(\mathcal{I})$. In the case of the vacuum state this is a necessary and sufficient condition to identify uniquely the one-parameter group of dilations as the modular group [18] mentioned above.

By a simple reparametrization of the unit-circle $S^1 \subset \mathbb{C}$ in terms of the conformal mapping $S^1 \ni z \longmapsto z^n$, $1 < n \in \mathbb{N}$, Schroer and Wiesbrock construct a geometrical state $\omega_2$ for the case $n = 2$. We shall henceforth refer to it as F-S state. This state is shown to be invariant under transformations of the form (4) for $n = 2$:

$$g_2(z) := \left(\frac{\alpha z^2 + \beta}{\bar{\beta} z^2 + \bar{\alpha}}\right)^{\frac{1}{2}} \qquad \begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix} \in PSU(1,1). \tag{9}$$

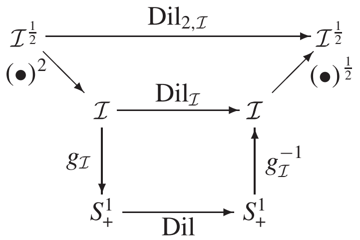
For intervals:

$$\mathcal{I}^{\frac{1}{2}} := \mathcal{I}_1 \cup \mathcal{I}_2 \qquad \mathcal{I}_i \xrightarrow{z^2} \mathcal{I} \qquad i = 1, 2 \tag{10}$$

the modified dilations act the following way:

$$\begin{aligned}\mathrm{Dil}_{2,\mathcal{I}}(t)\ (\bullet) &:= \ (\mathrm{Dil}_{\mathcal{I}}(t)\ (\bullet)^2)^{\frac{1}{2}}\\ &= \left(g_{\mathcal{I}}^{-1}\ \mathrm{Dil}(t)\ g_{\mathcal{I}}\ (\bullet)^2\right)^{\frac{1}{2}}\end{aligned} \tag{11}$$

(see the following diagram)




The F-S state $\omega_2$ may be defined in the following manner:

$$\omega_2(W(f)\ W(g)) := \omega_0\big(W\big(f_{\frac{1}{2}}\big)\ W\big(g_{\frac{1}{2}}\big)\big) \qquad W(f), W(g) \in \mathcal{W}(\tilde{\mathcal{I}}) \tag{12}$$

with $f_{\frac{1}{2}}(\bullet) := f\big((\bullet)^{\frac{1}{2}}\big)$, $\mathrm{supp}\big(f_{\frac{1}{2}}\big) \subset \mathcal{I}, \tilde{\mathcal{I}} \xrightarrow{z^2} \mathcal{I}$. It amounts to the following pointwise prescription for the current two-point function:

$$\omega_2(J(z)J(w)) := 2z\ 2w\ \omega_0(J(z^2)J(w^2)). \tag{13}$$

At this point, it is essential that the interval $\tilde{\mathcal{I}}$, i.e. localization region of the algebra $\mathcal{W}(\tilde{\mathcal{I}})$ in equation (12), does not contain opposite points $z, w \in S^1$, $\arg(z) - \arg(w) = 0, \bmod \left(\frac{2\pi}{2}\right)$. Otherwise, the F-S state becomes non-faithful as one can see easily by the following example of an operator $W \in \mathcal{W}(\tilde{\mathcal{I}})$:

$$W := 1 - W(g)W(f) \qquad f|_{\mathrm{supp}(f)} = -g|_{\mathrm{supp}(g)} \qquad \mathrm{supp}(f) = -\mathrm{supp}(g) \subset \tilde{\mathcal{I}}. \tag{14}$$

Localizing the algebra $\mathcal{W}$ in only one of the two intervals $\mathcal{I}_i \subset \mathcal{I}^{\frac{1}{2}}$, $i = 1, 2$, one is able to identify the modular group of the standard tuple $(\mathcal{W}(\mathcal{I}), \omega_2)$ ($\mathcal{I}$ stands for the chosen interval, i.e. $\mathcal{I}_1$ respectively $\mathcal{I}_2$) with the transformations in equation (11) by showing the KMS-property for the state $\omega_2$:[3]

$$\omega_2\left(W(f)\ Ad\left[U_{\mathrm{Dil}_{2,\mathcal{I}}(t)}\right](W(g))\right) \overset{\mathrm{KMS}}{=} \omega_2\big(Ad\left[U_{\mathrm{Dil}_{2,\mathcal{I}}(t+i)}\right](W(g))\ W(f)\big). \tag{15}$$

[3] Schroer and Wiesbrock essentially did the calculations in [1], whereas in [3] the correction, i.e. the limitation to only one of the two intervals as localization region was mentioned.

Using equation (12) this may be reduced to the vacuum case:

$$\omega_0\big(W\big(f_{\frac{1}{2}}\big)\, Ad\big[U_{\mathrm{Dil}_{\mathcal{I}}(t)}\big]\big(W\big(g_{\frac{1}{2}}\big)\big)\big) \overset{\mathrm{KMS}}{=} \omega_0\big(Ad\big[U_{\mathrm{Dil}_{\mathcal{I}}(t+i)}\big]\big(W\big(g_{\frac{1}{2}}\big)\big)\, W\big(f_{\frac{1}{2}}\big)\big). \tag{16}$$

The faithful F-S state $\omega_2$ on the algebra $\mathcal{W}(\mathcal{I})$ has a unique vector implementation $|\omega_2\rangle$ in the natural cone $\mathcal{P}_{|\omega_0\rangle}$ of the standard pair $(\mathcal{W}(\mathcal{I}), |\omega_0\rangle)$ [3, 16]:

$$\omega_2(W) = \langle\omega_2|W|\omega_2\rangle \qquad W \in \mathcal{W}(\mathcal{I}). \tag{17}$$

We want to point out in the following section that the above results of Schroer and Wiesbrock concerning the invariance of the modified states with respect to the modified $PSU(1,1)$ group and the KMS-property of these states with respect to the modified dilations are general properties for any $\mathrm{CFT}_2$ since the mentioned properties can, as we will show, be drawn through the substitutions $z \longmapsto z^n$, $\forall n \in \mathbb{N}$.

## 3. Extension to general rational conformal field theories

In the following we keep the pointwise prescription of fields. We generalize to arbitrary $n \in \mathbb{N}$, defining states $|\omega_n\rangle$ for local chiral primary fields $\phi(z)$ on $S^1$ by the identity:

$$\langle\omega_n|\prod_{k=1}^{l}\phi_k(z_k)|\omega_n\rangle := \prod_{s=1}^{l}\left(nz_s^{n-1}\right)^{\Delta_s}\langle\omega_0|\prod_{k=1}^{l}\phi_k\left(z_k^n\right)|\omega_0\rangle. \tag{18}$$

Using the Möbius invariance of the vacuum $|\omega_0\rangle$ one can show the invariance of $|\omega_n\rangle$ under transformations of the form (4) as follows:

$$\phi_i(z_i) \longrightarrow (\partial_z\{g_n(z)\}|_{z=z_i})^{\Delta_i}\phi_i(g_n(z_i))$$

$$\begin{aligned}
\langle\omega_n|\prod_{k=1}^{l}\phi_k(z_k)|\omega_n\rangle &\longrightarrow \prod_{i=1}^{l}(\partial_z\{g_n(z)\}|_{z=z_i})^{\Delta_i}\,\langle\omega_n|\prod_{j=1}^{l}\phi_j(g_n(z_j))|\omega_n\rangle\\
&\overset{\mathrm{eq}(18)}{=} \prod_{i=1}^{l}(\partial_z\{g_n(z)\}|_{z=z_i})^{\Delta_i}\prod_{j=1}^{l}(n\{g_n(z_j)\}^{n-1})^{\Delta_j}\,\langle\omega_0|\prod_{k=1}^{l}\phi_k(\{g_n(z_k)\}^n)|\omega_0\rangle\\
&\overset{\mathrm{eq}(4)}{=} \prod_{i=1}^{l}(\partial_z\{g_n(z)\}|_{z=z_i})^{\Delta_i}\prod_{j=1}^{l}(n\{g_n(z_j)\}^{n-1})^{\Delta_j}\,\langle\omega_0|\prod_{k=1}^{l}\phi_k\left(g\left(z_k^n\right)\right)|\omega_0\rangle\\
&\quad g \in PSU(1,1)\\
&= \prod_{i=1}^{l}(\partial_z\{g_n(z)\}|_{z=z_i})^{\Delta_i}\prod_{j=1}^{l}(n\{g_n(z_j)\}^{n-1})^{\Delta_j}\\
&\quad\times\prod_{r=1}^{l}\frac{1}{\left(\partial_z\{g(z)\}|_{z=z_r^n}\right)^{\Delta_r}}\,\langle\omega_0|\prod_{k=1}^{l}\phi_k\left(z_k^n\right)|\omega_0\rangle\\
&\overset{\mathrm{eq}(18)}{=} \langle\omega_n|\prod_{i=1}^{l}\phi_i(z_i)|\omega_n\rangle.
\end{aligned} \tag{19}$$

For the one-parameter group of modified dilations:

$$\begin{aligned}
\mathrm{Dil}_n(t)z &:= (\mathrm{Dil}(t)z^n)^{\frac{1}{n}}\\
&= \left(\frac{\mathrm{ch}(\pi t)z^n + \mathrm{sh}(\pi t)}{\mathrm{sh}(\pi t)z^n + \mathrm{ch}(\pi t)}\right)^{\frac{1}{n}}
\end{aligned} \tag{20}$$

one would like the following KMS-relation to be satisfied:

$$
\begin{aligned}
&\left(\partial_v\{\mathrm{Dil}_n(t)v\}|_{v=z_l}\right)^{\Delta_l} \langle\omega_n| \prod_{k=1}^{l-1} \phi_k(z_k)\, \phi_l(\mathrm{Dil}_n(t)z_l)|\omega_n\rangle \\
&\quad \overset{\mathrm{eq}(18)}{=} \prod_{i=1}^{l-1} \left(nz_i^{n-1}\right)^{\Delta_i} \left[(\partial_v\{\mathrm{Dil}_n(t)v\}|_{v=z_l})^{\Delta_l}(n\{\mathrm{Dil}_n(t)z_l\}^{n-1})^{\Delta_l}\right] \\
&\qquad \times \langle\omega_0| \prod_{k=1}^{l-1} \phi_k\left(z_k^n\right) \phi_l\left(\mathrm{Dil}(t)z_l^n\right)|\omega_0\rangle \\
&\quad = \prod_{i=1}^{l} \left(nz_i^{n-1}\right)^{\Delta_i} \left(\partial_v\{\mathrm{Dil}(t)v\}|_{v=z_l^n}\right)^{\Delta_l} \langle\omega_0| \prod_{k=1}^{l-1} \phi_k\left(z_k^n\right)\ \phi_l\left(\mathrm{Dil}(t)z_l^n\right)|\omega_0\rangle \\
&\quad \overset{!}{\underset{(\mathrm{KMS})}{=}} \prod_{i=1}^{l} \left(nz_i^{n-1}\right)^{\Delta_i} \left(\partial_v\{\mathrm{Dil}(t+i)v\}|_{v=z_l^n}\right)^{\Delta_l} \langle\omega_0|\phi_l\left(\mathrm{Dil}(t+i)z_l^n\right) \prod_{k=1}^{l-1} \phi_k\left(z_k^n\right)|\omega_0\rangle \\
&\quad \overset{\mathrm{eq}(18)}{=} \left(\partial_v\{\mathrm{Dil}_n(t+i)v\}|_{v=z_l}\right)^{\Delta_l} \langle\omega_n|\phi_l(\mathrm{Dil}_n(t+i)z_l) \prod_{k=1}^{l-1} \phi_k(z_k)|\omega_n\rangle.
\end{aligned} \tag{21}
$$

We have managed here to prove the KMS-condition for the chiral part of the correlation function in the modified ($|\omega_0\rangle \mapsto |\omega_n\rangle$) theory provided that the KMS-condition holds in the unmodified theory. But the latter is not precisely true. One picks, in general, a monodromy shift in the analytic continuation $t \mapsto t + \mathrm{i}s$, $s \in [0, 1]$,[4] which amounts to a full circle in complex space, returning to the same point on a different Riemann sheet. We may assume that the monodromy under consideration is diagonalized in a suitably chosen basis of conformal blocks and that the ensuing phase factors are compensated by the inverse phase factors of the anti-chiral block function (we assume the $\phi_{(i,j)}(z, \bar{z})$ to be scalar operators). The general situation is, what concerns the cancellation of phasefactors, faithfully represented by the simplified situation in the case of two-point functions. The (vacuum-) two-point functions are already defined by conformal invariance (and locality: $\bar{\Delta} = \Delta$) to be of the form [15]:

$$\langle\omega_0|\phi_1(z, \bar{z})\phi_2(w, \bar{w})|\omega_0\rangle = \frac{C_{12}}{|z - w|^{4\Delta}}. \tag{22}$$

This two-point function fulfils the KMS-condition with respect to the one-parameter group of dilations $\mathrm{Dil}(t) \in PSU(1, 1)$ (equation (6)) as one can show by direct calculation. In the case of arbitrary $n$-point functions of local conformal fields $\phi_{(i,j)}(z, \bar{z})$ the above argument which holds for a rational conformal field theory [15] reduces the analytic structure essentially to the $U(1)$-vertex form, i.e. equation (22). Since the KMS-property holds for equation (22) it follows for the $n$-point functions as well. Here it is important that one needs both the chiral and the anti-chiral parts.

One has the following KMS-condition for general local conformal fields:

$$
\begin{aligned}
&\left(\partial_v\{\mathrm{Dil}(t)v\}|_{v=z_l}\right)^{\Delta_{i_l}} \left(\partial_{\bar{u}}\{\mathrm{Dil}(-t)\bar{u}\}|_{\bar{u}=\bar{z}_l}\right)^{\Delta_{j_l}} \\
&\qquad \times \langle\omega_0| \prod_{k=1}^{l-1} \phi_{(i_k,j_k)}(z_k, \bar{z}_k)\, \phi_{(i_l,j_l)}(\mathrm{Dil}(t)z_l, \mathrm{Dil}(-t)\bar{z}_l)|\omega_0\rangle
\end{aligned}
$$

[4] In statistical physics one has the interval $[0, \beta]$, $\{\beta\}$ the inverse temperature, and the states are called $\beta$-KMS states [16, 17].

$$\stackrel{\text{KMS}}{=} \left(\partial_v\{\mathrm{Dil}(t+i)v\}|_{v=z_l}\right)^{\Delta_{i_l}} \left(\partial_{\bar{u}}\{\mathrm{Dil}(-t-i)\bar{u}\}|_{\bar{u}=\bar{z}_l}\right)^{\Delta_{j_l}} \times \langle\omega_0|\phi_{(i_l,j_l)}(\mathrm{Dil}(t+i)z_l, \mathrm{Dil}(-t-i)\bar{z}_l) \prod_{k=1}^{l-1}\phi_{(i_k,j_k)}(z_k,\bar{z}_k)\,|\omega_0\rangle. \tag{23}$$

A sufficient condition for locality is $\Delta_{i_p} = \Delta_{j_p}$, $p = 1, \ldots, l$.

Going now to the Schroer–Wiesbrock ansatz, one has the following picture on the chiral and anti-chiral sectors, respectively, and therefore the modified states $|\omega_{n,n}\rangle$:

$$\langle\omega_{n,n}|\prod_{k=1}^{l}\phi_{(i_k,j_k)}(z_k,\bar{z}_k)|\omega_{n,n}\rangle := \prod_{s=1}^{l}\left(nz_s^{n-1}\right)^{\Delta_{i_s}} \prod_{q=1}^{l}\left(n\bar{z}_q^{n-1}\right)^{\Delta_{j_q}} \langle\omega_0|\prod_{k=1}^{l}\phi_{(i_k,j_k)}\left(z_k^n,\bar{z}_k^n\right)|\omega_0\rangle. \tag{24}$$

Again, the KMS-condition with respect to $\mathrm{Dil}_n(t)z := (\mathrm{Dil}(t)z^n)^{\frac{1}{n}}$ (equation (20)) on both sectors 'goes through covariantly' as in equation (21).

An illustration of what has been said above in a non-trivial setting (the $U(1)$-current algebra is a quasifree theory [8]) is provided by the theory of non-Abelian currents [15]. The central relation is the current–current commutation relation (Kac–Moody algebra, with the circle as base space):

$$[J^a(z), J^b(w)] = \mathrm{i}f^{abc}J^c(z)\delta(z-w) - k\delta^{ab}\partial_z\delta(z-w). \tag{25}$$

This relation allows it to calculate the $m$-point correlation function recursively by using the $m$-1- and $m$-2-point function:

$$\langle\omega_0|\prod_{i=1}^{m}J^{a_i}(z_i)|\omega_0\rangle \stackrel{\text{eq(25)}}{=} \sum_{j=2}^{m}\frac{k\delta^{a_1a_j}}{(z_1-z_j)^2}\langle\omega_0|\prod_{\substack{k=2\\k\neq j}}^{m}J^{a_k}(z_k)|\omega_0\rangle + \sum_{j=2}^{m}\frac{\mathrm{i}f^{a_1a_jd}}{(z_1-z_j)}\langle\omega_0|\prod_{k=2}^{j-1}J^{a_k}(z_k)J^d(z_j)\prod_{l=j+1}^{m}J^{a_l}(z_l)|\omega_0\rangle. \tag{26}$$

One can, therefore, verify the KMS-property inductively since the two-point function has the KMS-property. For the $|\omega_n\rangle$ state one gets the following recurrence relation:

$$\langle\omega_n|\prod_{i=1}^{m}J^{a_i}(z_i)|\omega_n\rangle \stackrel{\text{eq(18)}}{=} \sum_{j=2}^{m}\frac{\left(nz_1^{n-1}\right)\left(nz_j^{n-1}\right)k\delta^{a_1a_j}}{\left(z_1^n-z_j^n\right)^2}\langle\omega_n|\prod_{\substack{k=2\\k\neq j}}^{m}J^{a_k}(z_k)|\omega_n\rangle + \sum_{j=2}^{m}\frac{\left(nz_1^{n-1}\right)\mathrm{i}f^{a_1a_jd}}{\left(z_1^n-z_j^n\right)}\langle\omega_n|\prod_{k=2}^{j-1}J^{a_k}(z_k)J^d(z_j)\prod_{l=j+1}^{m}J^{a_l}(z_l)|\omega_n\rangle.$$

The KMS-condition with respect to $\mathrm{Dil}_n(t)z := (\mathrm{Dil}(t)z^n)^{\frac{1}{n}}$ (equation (20)) can be shown inductively as well, since the substitution $z \longmapsto z^n$ goes through covariantly and therefore the KMS-property for the vacuum carries over to the new state.

For the identification with the modular group mentioned above one needs von Neumann algebras. It is a subtle problem to proceed from smeared unbounded field operators to local algebras of bounded operators. By using bounded functions of unbounded local operators as in the case of the $U(1)$-current algebra one has to ensure locality which might not be conserved by this mapping. Bisognano and Wichmann gave certain sufficient conditions to identify the von Neumann algebras to which the (Wightman-) field algebras are affiliated [4, 17]. Generally it is a non-trivial task to verify these conditions. In the case of non-Abelian currents, this is possible and one can therefore speak of von Neumann algebras, respectively the associated modular group.

## 4. The split-property and doubly-localized algebras

We have seen that the localization region of the algebras has to exclude opposite points to guarantee the faithfulness of the F-S state $\omega_n$. In the following we will stick to the case $n = 2$ for simplicity. Since both intervals $\mathcal{I}_{1,2}$ (equation (10)) are allowed for localizing the algebra it seems natural to have a closer look at the two-interval, i.e. multilocalized, algebra $\mathcal{W}(\mathcal{I}_1) \vee \mathcal{W}(\mathcal{I}_2) = \mathcal{W}(\mathcal{I}_1 \cup \mathcal{I}_2) =: \mathcal{W}(\mathcal{I}^{\frac{1}{2}})$. The demand for faithfulness prohibits the direct use of the F-S state $\omega_2$. There is, however, a way to enforce faithfulness by applying the split-property [16, 9][5].

For two spacelike regions, i.e. disjoint intervals, the split-property implies the following:

$$\mathcal{W}(\mathcal{I}_1) \vee \mathcal{W}(\mathcal{I}_2) \simeq \mathcal{W}(\mathcal{I}_1) \otimes \mathcal{W}(\mathcal{I}_2). \tag{27}$$

This reflects in a certain sense the statistical independence of the algebras. The state $\omega_2$, faithful and normal over either interval $\mathcal{I}_{1,2}$ can be extended to a product state, faithful and normal over $\mathcal{W}(\mathcal{I}^{\frac{1}{2}})$, i.e.,

$$\omega_2^p(WV) = \omega_2(W)\omega_2(V) \qquad W \in \mathcal{W}(\mathcal{I}_1), \quad V \in \mathcal{W}(\mathcal{I}_2). \tag{28}$$

The modular group also splits and one can show the following theorem:

**Theorem.** *The modular group $\sigma^t_{\omega_2^p}$ of the faithful product state $\omega_2^p$ on the algebra $\mathcal{W}(\mathcal{I}^{\frac{1}{2}})$ is given by the geometric action of* $\mathrm{Dil}_2(t)$. *Moreover, the unitary implementer $U_{\mathrm{Dil}_2(t)}$ whose infinitesimal generator is a linear combination of $L_{\pm 2}$:*

$$\sigma^t_{\omega_2^p} = Ad\left[U_{\mathrm{Dil}_2(t)}\right] \tag{29}$$

*is the $\Delta^{it}_{\omega_2^p}$ modular object of the state vector $|\eta_2\rangle$ which represents the faithful product state $\omega_2^p$ in the positive cone of $(\mathcal{W}(\mathcal{I}^{\frac{1}{2}}), |\omega_0\rangle)$.*

Since the arguments are entirely similar to those which demonstrated that $U_{\mathrm{Dil}_2(t)}$ was the $\Delta^{it}$ modular group of the faithful state $\omega_2$ on either $\mathcal{W}(\mathcal{I}_1)$ or $\mathcal{W}(\mathcal{I}_2)$, we omit the details. The crucial point is the invariance:

$$\begin{aligned}\omega_2^p\left(Ad\left[U_{\mathrm{Dil}_2(t)}\right](WV)\right) &= \omega_2^p\left(Ad\left[U_{\mathrm{Dil}_2(t)}\right](W)\ Ad\left[U_{\mathrm{Dil}_2(t)}\right](V)\right)\\ &= \omega_2\big(Ad\big[U_{\mathrm{Dil}_2(t)}\big](W)\big)\ \omega_2\big(Ad\big[U_{\mathrm{Dil}_2(t)}\big](V)\big)\\ &= \omega_2(W)\ \omega_2(V)\\ &= \omega_2^p(WV)\end{aligned} \tag{30}$$

which uses the previously established invariance of $\omega_2$. The aforementioned lack of faithfulness of the state $\omega_2$ on the multi-interval algebra $\mathcal{W}(\mathcal{I}^{\frac{1}{2}})$ is related to the geometric nature of the double interval modular group $\sigma^t_{\omega_2^p}$.

The remarkable fact that the pairs $(\mathcal{W}(\mathcal{I}_i), \omega_2), i = 1, 2, (\mathcal{W}(\mathcal{I}^{\frac{1}{2}}), \omega_2 \times \omega_2)$ share the same modular group action and (in the appropriate positive cones) have the same implementing modular unitaries is also related to this. The factorization of the $\omega_0$ vacuum would not lead to such a situation.

This result raises the question whether both modular objects of the double interval situation $(\mathcal{W}(\mathcal{I}^{\frac{1}{2}}), |\eta_2\rangle)$ can be geometric. Formally, the candidate for a geometric $J$ is the 'rotated' TCP transformation $z \to -\bar{z}$ which maps, for instance, the first quarter-circle $S^1_1 := \left[0, \frac{\pi}{2}\right]$ to

[5] Another way might be the use of a (minimal) projector $E$ implying the faithfulness of $\omega_2$ on the reduced algebra $E\mathcal{W}E$.

the second quarter-circle $S_2^1 := \left[\frac{\pi}{2}, \pi\right]$, and the third quarter-circle $S_3^1$ to the fourth $S_4^1$ [6] (this transformation has the same geometric effect as the analytically continued $\Delta_{\omega_2^p}^{\frac{1}{2}}$). However, the replacement of Haag duality by an inclusion:

$$\mathcal{W}((\mathcal{I}_1 \cup \mathcal{I}_2)') \subset \mathcal{W}(\mathcal{I}_1 \cup \mathcal{I}_2)' \tag{31}$$

and its explanation in terms of superselection sectors shows that the true modular involution has, in addition to the geometric part, an algebraic modification. A calculation of $J$ for Abelian current models seems to be feasible. Via the inverse algebraic lightfront holography [3] these geometric chiral transformations correspond to fuzzy symmetries in the original net.

## 5. Summary

It was shown that $\mathrm{Dil}_n$ (having $2n$ fixpoints instead of two as Dil) is the modular group of the standard tuple $(\mathcal{W}(\mathcal{I}), \omega_n)$. The interval $\mathcal{I}$ is forbidden to contain points $z, w \in S^1, \arg(z) - \arg(w) = 0, \mathrm{mod}\left(\frac{2\pi}{n}\right)$.

With regard to the modular origin of the Witt–Virasoro basis it is sufficient to extend the construction up to the $n = 2$ case due to relation (1).

In the case of multi-interval, i.e. multilocalized Weyl algebras (equation (27)), transformations (29) can be identified with the modular group of the tuple $(\mathcal{W}(\mathcal{I}^{\frac{1}{2}}), |\eta_2\rangle)$. Going to higher $n$ by using the split-property is not totally straightforward.

Following the programme of algebraic quantum field theory this result underlines the special rôle played by the modular theory in general local quantum field theory. Schroer and Wiesbrock, Schroer and Fassarella [3] suggest the use of modular theory as a tool to explore 'fuzzy' symmetries, i.e. symmetries which do not originate from the classical Noether setting. They also propose to investigate the relation of their findings with the notion of *half-sided modular inclusion (intersections)* [2, 5, 13]. See [6, 14] for a recent account and further references to the results in [2, 5, 13].

Investigating the latter is beyond the scope of the present work. Here, we show that a new ansatz of Schroer and Wiesbrock (corrected in [3]) can be dealt with from a more general point of view, since the KMS-property of the vacuum state (a general property for local fields in the vacuum setting) carries over covariantly to the set of new F-S states. The argument for the general validity of the KMS-property given above is true for rational conformal field theories.

Multilocalized algebras can be defined by using the split-property which implies the faithful states to be a product of the new constructed F-S states. The modular group naturally splits into a product of two copies of the modified dilations whereas the modular conjugation needs some further investigation.

*Note added.* We are indebted to Professor Schroer for pointing out to us the incorrectness of the implicitly assumed faithfulness of the new states with respect to multilocalized algebras in [1]. Section 4 is based on notes by Professor Schroer.

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[6] The quarter-circle algebras $\mathcal{W}(S_{1,3}^1)$ related to the algebra localized on the upper semi-circle $\mathcal{W}(S_+^1)$ provide a natural example.

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